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NUMERICAL AND EXPERIMENTAL ANALYSES OF HOPF BIFURCATIONS IN A LOCALLY EXPANDED CHANNEL

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Key words: Hopf bifurcations, CFD.

Abstract

This article discusses Hopf bifurcations in fluid flow through a locally expanded channel. The first part presents the simulation model and computation results that investigate the possibility of oscillatory bifurcations in the analyzed system at given geometry configurations and parameters. Simulations were performed with the application of the Finite Volume Procedure in the Multi Flower 2D non-commercial package. The second part attempts to supplement simulation results with the use of FLUENT and FlowWorks commercial applications. The last part of the paper describes a laboratory experiment validating the results of numerical computations at the qualitative and, partly, the quantitative level. The described experiments investigate the usefulness of CFD applications and simulation techniques in predicting and analyzing bifurcations under real flow conditions.

ANALIZA NUMERYCZNA I EKSPERYMENTALNA BIFURKACJI HOPFA W KANALE Z LOKALNYM ROZSZERZENIEM

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Słowa kluczowe: bifurkacje Hopfa, CFD.

Abstrakt

Artykuł zawiera rozważania dotyczące występowania bifurkacji Hopfa w przepływie płynu przez kanał z lokalnym, występującym tylko na pewnym odcinku, rozszerzeniem. W pierwszej części przedstawiono model symulacyjny oraz wyniki obliczeń, wskazujące na możliwość występowania w rozważanym układzie bifurkacji oscylacyjnych przy pewnych konfiguracjach geometrii i parametrów. Symulacje wykonano za pomocą niekomercyjnego pakietu Multi Flower 2D, opartego na metodzie objętości skończonych. W części drugiej podjęto próbę uzupełnienia wyników symulacji z zastosowaniem komercyjnych pakietów FLUENT oraz FlowWorks. Ostatnia część artykułu przedstawia oryginalny eksperyment laboratoryjny, potwierdzający na poziomie jakościowym i częściowo ilościowym wyniki obliczeń numerycznych. Opisane w pracy badania przedstawiają możliwości wykorzystywania aplikacji CFD i technik symulacyjnych do przewidywania bądź analizowania zjawisk bifurkacyjnych w rzeczywistych układach przepływowych.

Introduction

Nonlinear differential equations can be used to describe various physical phenomena in nature. The equilibrium stability of nonlinear systems has a local character in reference to initial conditions as well as disturbances. The above implies that the effect is not proportional to the parameter describing the cause. Even minor changes in the initial conditions of a system may produce large variations in the long-term behavior of the system. This phenomenon has been referred to as the "butterfly effect" following the publication of Edward Lorentz's article entitled "Can the flap of butterfly's wing stir up a tornado in Texas?". Research efforts are made to investigate whether the processes observed in various systems (physical, chemical, biological, economic, demographic, etc.) are steady, i.e. minimally sensitive to perturbations, and predictable. Due to unpredictable events, a system's equilibrium may be altered to reach an unacceptable or even an undesirable state. Many catastrophes occur when a system shifts from a static to a dynamic equilibrium, such as buckling of rod structures, vibration of suspended bridges or the flatter of airplane wings. Those changes and transitions are related to bifurcation.

In mathematical terms, bifurcation occurs when a minor change in parameters causes the properties of a model to change (MARDSEN 1976, HARB 1996, BADUR, SOBIESKI 2001, LEINE 2006, RODRIGUES 2007). In practice, it implies a splitting of the equilibrium solution branch when a given active (bifurcation) parameter reaches its critical value. In mechanics, bifurcation is defined as the emergence of new momentum, heat and mass transfer components as external conditions change. Two principal bifurcation conditions have been determined in both mechanical and thermal systems (BADUR, SOBIESKI 2001, SOBIESKI 2006):

– the existence of a dominant flow direction, i.e. a direction in which the values of parameters such as velocity, displacement or temperature gradient are much higher than in the remaining (perpendicular) directions,

– the existence of a "free area" perpendicular to the dominant direction, i.e. the possibility of solution branching.

An example of bifurcation in fluid mechanics is thin, stationary fluid flow through an expanded (locally or permanently) channel. The expansion creates a "free area", and the convective part of the momentum flux is responsible for the loss of stability. The active parameter in this case is the Mach number or the Reynolds number.

Other types of bifurcation behavior are also encountered in flow systems, including Taylor-Couette flow between two rotating cylinders (DOMAŃSKI 2006) or Rayleigh-Bènard cells in selected types of convective flow (Elmer Tutorials 2007).

Two types of bifurcations are possible in the analyzed expanded (locally and permanently) channel. The first type is divergent (stationary) bifurcation in which the system initially undergoes a sudden change, after which it stabilizes and takes up a new, stationary form. The second type is oscillatory bifurcation where the permanent solution loses stability and is replaced by a periodic solution that develops with an active parameter increase in the supercritical area (KURNIK 1997, SOBIESKI 2006). In real physical systems, the above corresponds to the loss of equilibrium point stability and the occurrence of self-excited vibrations. Cases with periodic vibrations are referred to as Hopf bifurcations.

There are very few published sources investigating flow structures in locally expanded channels (expanded along a given section). One of the most noteworthy examples is the work of MULLIN et al. (2003) which describes the experiment and presents the results of numerical simulations. The authors of the study identified two basic types of structures: symmetrical and asymmetrical, and concluded that the type of bifurcation is determined mostly by the system's geometry relations. Similar studies have been carried out in reference to permanently expanded channels (an expansion from a set point to the end of the investigated flow area) (BATTAGLIA et al. 1997 MANICA BORTOLI DE 2003), but the conducted experiments were purely simulational. There are several experimental studies investigating flows through permanently expanded channels (ESCUDIER 2002, POOLE 2005, CHIANG et al. 2001), but they make a partial reference to non-Newtonian fluids and flows at a low Reynolds number.

The above cited studies have been carried out based on a classical approach: an experiment is performed and further attempts are made to repeat it virtually with the involvement of a simulation model. The author of this study attempted to reverse this classical approach. The study began with the development of a simulation model and the relevant assumptions. A test stand was then set up based on the results of the simulation phase. This approach naturally gave rise to two key experimental objectives. The first goal was to model the previously investigated bifurcation phenomena observed in locally expanded channels. The second objective was to determine to what extent Hopf bifurcations can be predicted in a real system based on a simulation model.

Computer model

The geometry of the investigated flow structure is presented in Figure 1. The system comprised a plain channel with a symmetrical, local expansion in the center.

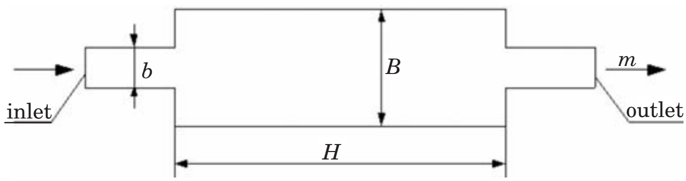

Fig. 1. Flow system geometry

The basic model was defined and the following assumptions were made prior to modeling:

– the experiment analyzes a single-component flow of a medium whose properties resemble those of water (the liquid cannot be directly defined in the applied computational program due to the need to account for the compressible fluid equation),

– the Mach number should be less than 0.3 to maintain medium parameters at the level of near real-event parameters (which follows directly from the previous assumption),

– computation will take place in two-dimensional space (DYBAN 1971, BADUR, SOBIESKI 2001, ŚWIĄTECKI 2004, CUDAKIEWICZ 2005, SOBIESKI 2006),

– the effect of turbulence and viscosity is considered to be negligibly small (BADUR 1999, ŚWIĄTECKI 2004, CUDAKIEWICZ 2005, SOBIESKI 2006),

– outlet pressure is constant and equal to atmospheric pressure at 1013 [hPa],

– static input pressure is constant and equal to 1025 [hPa],

– inlet velocity is controlled by modifying total inlet pressure values,

– inlet and outlet width is identical and constant at 25 [mm],

– total channel length is constant at 280 mm [mm],

– total expansion length H is constant at 200 [mm],

– basic channel width in the expanded section is 50 [mm],

– the result is positive, i.e. a Hopf bifurcation occurs, when the flow is asymmetrical and cyclical changes in flow type are observed. The result is negative when the flux impinges on a side wall and becomes detached only at the outlet.

Following system analyses and preliminary simulations, the following numerical parameters were adopted in the study (PUCHALSKI 2008):

– type of grid: structural, total number of cells: 48 000,
– convective flux reconstruction method: TVD (Total Variation Diminishing),
– time-stepping scheme: implicit, with constant time step for all cells,
– CFL number (Courant-Friedrich-Levy condition): 10,
– minimum number of iterations: 20 000.

The flow simulation model was developed with the use of the Multi Flower 2D v. 3.0.6 non-commercial package (SOBIESKI 2006, SOBIESKI 2008). The package combines applications for developing computer simulations in compressible fluid mechanics and features the following options: modeling two-dimensional flows with every type of geometry relations (internal and external), modeling stationary and non-stationary flows, modeling subsonic, supersonic and transonic flows, modeling multicomponent flows. The Multi Flower 2D package relies on the Finite Volume Procedure (FV) and the Mixture Model. The FV procedure supports multiple balancing of different values (mass, momentum, energy, etc.) in every computational cell and at every time step.

The general system of balance equations found in the Multi Flower 2D solver is as follows (BADUR, SOBIESKI 2001, SOBIESKI 2006):

– mass balance: $\frac{\partial}{\partial t}\rho + \nabla(\rho\vec{v}) = 0$ (1)

– momentum balance: $\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla(\rho\vec{v}\otimes\vec{v}) = \nabla(-p\overleftrightarrow{I} + \overleftrightarrow{\tau}) + \rho\vec{s}_b$ (2)

– energy balance: $\frac{\partial}{\partial t}(\rho e) + \nabla(\rho e\vec{v}) = \nabla((-p\overleftrightarrow{I} + \overleftrightarrow{\tau})\,\vec{v} + \vec{q}) + \rho s_e$ (3)

– component balance: $\frac{\partial}{\partial t}(\rho Y_k) + \nabla(\rho Y_k\vec{v}) = \nabla(\vec{J}_k) + \rho s_k$ (4)

where: ρ – density of mixture [kg m^{-3}], $\vec{v}$ – average velocity of mixture [m s^{-1}], $\overleftrightarrow{\tau}$ – total stress tensor [Pa], $\vec{s}_b$ – source of mass forces [N m^{-3}], e – total energy [J], p – pressure of mixture [Pa], $\overleftrightarrow{I}$ – unit tensor [-], $\vec{q}$ – total heat flux [J/(m^{2}s^{-1})], s_e – source of energy [J/(m^{3}s^{-1})], Y_k – mass share of the kth component [-], $\vec{J}_k$ – total diffusion flux [mol/(m^{2}s^{-1})], s_k – mass source [kg/m^{3}s^{-1}], n_s – number of mixture components. The subscript in the equation (4) may be from 1 to n_s.

Equations (1)–(4) present the Mixture Model (or the Homogenous Model) for describing homogenous mixes of any number of different phases: gases, fluids and solids. In this model, all phases are regarded as a mixture and have a single balance equation system. The mixture is described with the use of Euler approach (SOBIESKI 2008). The discussed simulations did not require the use of the multicomponent flow modeling option.

The Multi Flower 2D package solver does not contain all elements featured in equations (1)–(4). It does not list stress tensors or source terms. Despite those simplifications (and taking into account the previous assumptions), the Multi Flower 2D package seems to be an appropriate tool for the needs of this study. Its usefulness has been validated in an earlier study on the modeling of bifurcations in a closed-off plain channel based on a previous experiment carried out by DYBAN (1971) and described by BADUR and SOBIESKI (2001). A high level of qualitative consistence and a satisfactory level of quantitative consistence were reported in that study. The suitability of the Multi Flower 2D package for modeling non-stationary processes with bifurcations has also been confirmed by other authors (ŚWIĄTECKI 2004, CUDAKIEWICZ 2005, PUCHALSKI 2008).

Simulation results

The first phase of the numerical simulation process involved the determination of an area with Hopf bifurcations. Such bifurcations were observed in geometry relations later referred to as the basic model at a total pressure of 105 000 [Pa] (Fig. 2). The maximum computational velocity was 3.85 [m s^{-1}], and the value of the Mach number was below the set limit of 0.3. The first phase ended with the determination of the parameter range for further computations, i.e. channel width B of 40 to 60 [mm], at intervals of 5 [mm], and a total input pressure of 103 000 to 107 000 [Pa], at intervals of 250 [Pa].

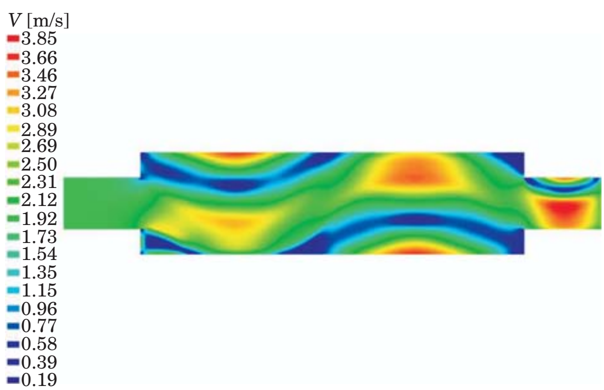

Fig. 2. Basic flow with Hopf bifurcations – total velocity distribution

The second simulation series began with channel width B = 50 [mm] to identify the range at which bifurcations occur. A positive result was reported in a pressure range of 104 500 to 105 000 [Pa]. Width B was changed to 45 [mm] and subsequent efforts were made to determine the range at which Hopf bifurcations appear. This procedure was repeated for all of the set channel widths. In the case with the lowest analyzed width, a positive result was noted within the range of 104 000 to 120 000 [Pa], but some of the results were disregarded in view of the assumed Mach number; therefore, the maximum total pressure was deemed as correct at 10 600 [Pa].

The Mach number minimally exceeded the threshold value at channel width B = 45 [mm] and a pressure of 104 250 and 104 500 [Pa]. Those results were not rejected, but were regarded as less reliable. A positive result was not reported for width B = 60 [mm] at any range of input pressure values.

At channel width B = 55 [mm] and a pressure of around 103 000 [Pa], a new type of flow was observed, but it was not classified as bifurcation and was subsequently disregarded. The results of this experimental phase are presented in Table 1. Basic model results are highlighted with a different background color.

Table 1

Hopf bifurcation areas

B [mm]	Total pressure [hPa]												
	103	103.25	103.5	103.75	104	104.25	104.5	104.75	105	105.25	105.5	105.75	106
40				X	V	V						V	V
45					X	V	V		V	V	V	V	X
50	X				X	X	V	X	V	X		X	
55		X	X	X	X			X	X	X	X	X	
60								X	X				

The objective of the next stage of the study was to verify whether the observed bifurcations were Hopf bifurcations. For this purpose, total pressure values at a characteristic flow point were registered during the computational process. The selected point was situated minimally outside the flow axis at an adequate distance from the beginning of the expansion (at around 1/4 length of the entire expanded section). Calculations were repeated for B = 50 [mm] and a total inlet pressure of 105 000 [Pa] because oscillating flow characteristics were clearly manifested in the investigated case. The pressure change diagram is shown in Figure 3. The regularity of pressure changes after the equations have been expanded proves that the observed process is a case of oscillatory bifurcations.

Flow periodicity is also manifested in the convergence diagram (Fig. 4). The system solution follows the pattern of a periodically changing flow throughout the computation process, but it never reaches a stationary state. It should be noted that the frequency of the residue cycle is twice higher than the frequency of pressure changes. The above indicates that the computational process expands in the same way after every "reversal" of the flow structure (which takes place twice in each bifurcation cycle).

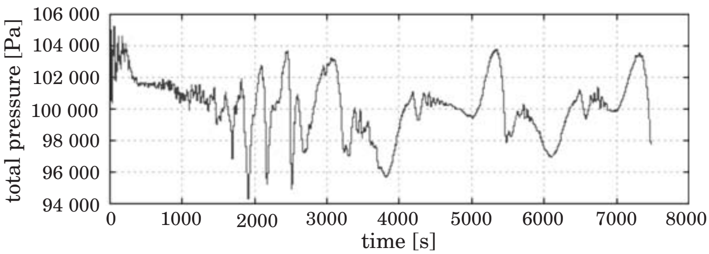

Fig. 3. Diagram of changes in total pressure in a selected cell

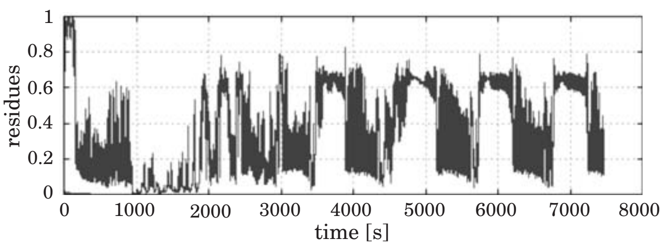

Fig. 4. Convergence process for the basic model

Simulations of prebifurcation flow

Attempts were made to replicate the above results with the use of different software. Simulation models were developed in the FLUENT (based on the Finite Volume Procedure) and the FlowWorks (based on the Finite Elements Procedure) commercial packages. A 3D model was applied in both cases to verify the effect of the number of parameters on computation results. Two computational grids (30 200 and 244 000 cells) and several initial conditions were analyzed in the FLUENT package. Only the basic model with around 25 000 grid cells was analyzed in the FlowWorks package. It should be noted that the authors were unable to precisely replicate the model generated in the Multi Flower 2D package in either of the commercial programs. The reported deviations concerned the definition of the working medium. An incompressible medium was applied in FLUENT and FlowWorks packages, while the Multi Flower 2D relied on a compressible medium described by the ideal gas law.

In general, it was found that the computational algorithms in the FLUENT and FlowWorks packages, which are highly complex and guarantee computational stability for a very broad range of values, suppress bifurcation and produce results that are completely inconsistent with expectations. The results of the basic model simulation, achieved with the use of the above commercial packages, are presented in Figures 5 and 6. Special attention should be paid to the range of total velocity values which are similar in Multi Flower 2D and FlowWorks packages, but are significantly different in the FLUENT application. The causes of the above were not identified.

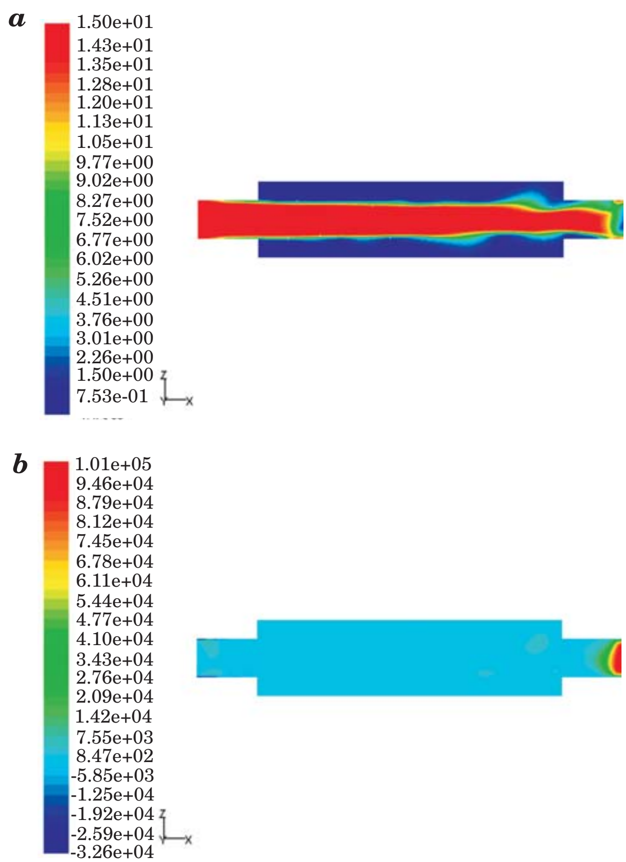

Fig. 5. Distribution of total velocity and static pressure produced in the FLUENT package for the basic model

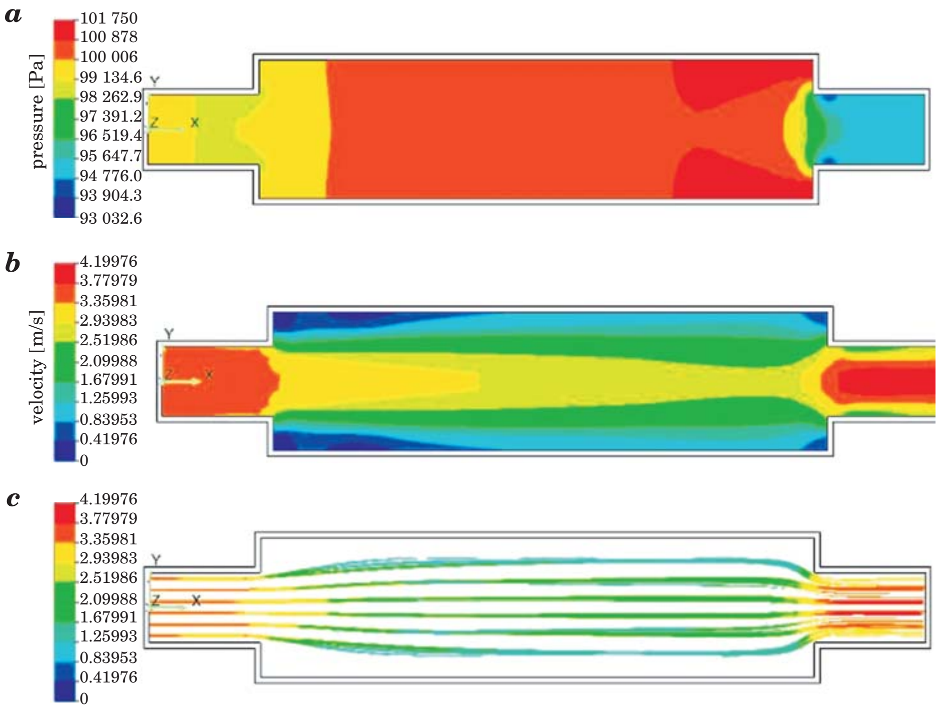

Fig. 6. Distribution of static pressure, total velocity and stream line course observed in the FlowWorks package for the basic model

Laboratory experiment

Since previous experiments were highly general in nature and were carried out at the qualitative level, the author of this study set out to investigate whether simulation results could be applied to predict bifurcations in real systems. A test stand was set up to determine the above. The author assumed that the key factor determining bifurcations was the geometry of the flow system, namely the b/B width ratio. Secondary importance was attached to the type of medium and its parameters.

The set-up of the test stand is presented in Figure 7. The main part comprises two plexiglass panels enclosing partially mobile metal walls. Channel dimensions conform to the dimensions of the previously described basic model. Channel width (in the third spatial dimension) was 20 [mm], and it was deemed sufficient to minimize flow disturbances caused by the viscosity effect. A greater width was not applied to ensure adequate flow intensity. The test stand was supplied from the local water mains.

Fig. 7. Laboratory test stand

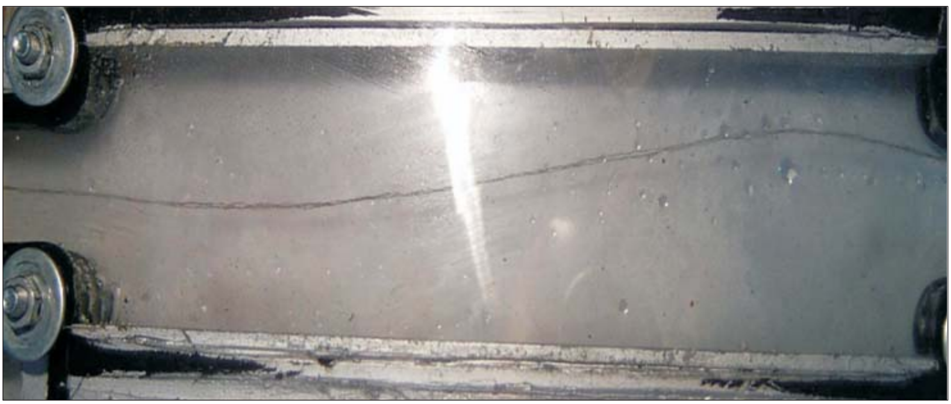

Fig. 8. Flow visualization at an intensity level of 0.72 [l s^{-1}]

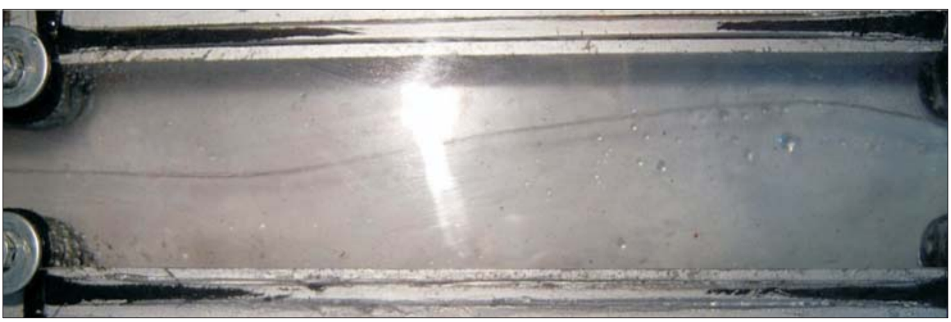

Fig. 9. Flow visualization at an intensity level of 0.97 [l s^{-1}]

The behavior of the water jet was observed upon reaching the expanded channel section at different flow intensity values. A dye was initially introduced into the system together with the liquid for the purpose of visualization, but it failed to achieve the set objective in turbulent flows. The dye became mixed with water too early, and it did not reflect the movement of the water jet. As part of the second attempt to visualize the flow structure, a thin thread was introduced into the system and attached symmetrically at the channel inlet. The experiment was performed at four flow intensity levels Q = 0.08, 0.33, 0.72, 0.97 [l s^{-1}] to produce the following average flow velocity values at the inlet c = 0.16, 0.65, 1.44, 1.93 [m s^{-1}]. In the first two instances, flow intensity was too low and the thread rested loosely in the channel. At higher flow intensity levels, the thread clearly oscillated and took up the anticipated shape. The above is presented in Figures 8 and 9. Flow velocity was roughly consistent with the values computed in the Multi Flower 2D simulation. Higher intensity flows could not be investigated for technical reasons which prevented the determination of the upper intensity limit at which oscillations occur. It should also be noted that jet oscillations were clearly manifested in the experiment, but the flow was not ideally symmetrically reversed in relation to the axis. The above could be attributed to insufficient accuracy or the manner of thread attachment: the thread had a greater tendency to become reversed in a single direction. The needle placed inside the channel was not ideally centered which also affected measurement precision. The frequency of flow reversal could not be determined with a high degree of accuracy, but according to estimates, it ranged from around 0.2 [Hz] to around 1.5 [Hz]. The above observation is not consistent with the results of the numerical simulation where the computational time per cycle was much longer.

Conclusions

The conducted experiment enabled to formulate the following final conclusions:

– bifurcation flows (including Hopf bifurcations) can be predicted and analyzed with the use of simulation techniques developed as part of Numerical Fluid Mechanics. In view of scant publications addressing this problem, the reported results render this experiment a success;

– the available numerical codes, FLUENT and FlowWorks, do not feature solvents for analyzing bifurcation flows. The above could be due to the high level of complexity of the applied computational algorithms which maximally protect the computational process from the loss of stability. The solver of the Multi Flower 2D package is less stable and less versatile, but it is incapable of suppressing system disturbances with equal efficiency, thus supporting the propagation of bifurcation phenomena. This theory has not been validated, but it has been postulated by the author as a possible explanation;

– the experiment validated the assumption that the *b*/*B* width ratio is the key factor responsible for bifurcations in expanded channels. The fact that bifurcations were produced in a real system designed solely in view of previous numerical analyses is a success. Although it was impossible to precisely verify each computational case, the validation of the basic model proved the efficiency of simulation methods and could be a venture point for future research. Follow-up experiments are needed to determine the range of values at which Hopf bifurcations occur in a real system and to compare this range of values with the results of simulations. A study to determine the effect of expansion length *H* (or dimensionless quantity *H*/*B*) on the nature of non-stationary phenomena would also greatly contribute to our understanding of the problem;

– the experiment confirmed the assumption that the type of medium plays a secondary role in the bifurcation process. These findings are supported by the experiment involving a completely different medium than that applied in the simulations (due to the limitations of the simulation program) which produced similar results;

– a satisfactory quantitative conformity of flow parameters was nor achieved. In a real system, bifurcations were observed at a much higher inlet pressure than in numerical simulations (where even minor pressure differences resulted in bifurcations). The above could be due to the specific nature of the applied mathematical model, in particular the equation of state which is more suitable for gas than liquid. This assumption is validated by velocity values which, despite significant differences in inlet pressure, are similar. Total velocity in basic model simulations ranged from 0.19 to 3.85 [m s^{-1}] regardless of the place in space (for the time horizon presented in Fig. 2). In a real system, similar flow values were noted at average inlet velocity equal to 0.72 and 0.98 [m s^{-1}]. The observed velocity values cannot be directly compared due to the absence of data on minimum and maximum velocity in a real system (whose determination requires the use of specialist measurements, such as the PIV method) and changes in these values over time. It is possible that the temporal moment presented in Figure 2 is not the best indicator of the velocity field (minimum and maximum velocity values varied for different temporal moments);

– the analysis of the effect of flow velocity, partially discussed in the previous point, validates reference data according to which in addition to geometry relations, velocity (the Reynolds number in the dimensionless form) is the second most important factor (active parameter) affecting the occurrence and character of bifurcations.

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